SET THEORY AND FORCING

Spring 2018 MATH 574 Anush Tserunyan

This course is an introduction to the Zermelo–Fraenkel set theory with Choice (ZFC): an axiomatic system/theory generally accepted as the foundations of mathematics. As one expects, the theory that captures most of mathematics is very complicated and, unsurprisingly, the study of its models and the boundaries of what it can and cannot prove has grown into a complex and multifaceted mathematical discipline, which often has combinatorial flavor.

The basics and absoluteness. The course starts with the basics of set theory, namely: the axioms of ZFC, ordinals, transfinite induction, cardinals, and some cardinal arithmetic—what arguably every mathematician should know. We will also discuss *Shoenfield's Absoluteness theorem*: if you manage to prove a statement (of a certain general form) in analysis/dynamical systems/probability theory using an extra ZFC-independent assumption (e.g. the Continuum Hypothesis), then Shoenfield's Absoluteness implies that there is a proof of your statement without this extra hypothesis, thus yielding an unconditional theorem.

Gödel's universe L. This is the smallest submodel of any model of ZFC. What L is to a model of ZFC is what the prime subfield is to a given field. In L, the Continuum Hypothesis holds, but many desirable statements fail: for example, you can start with a Borel subset of $\mathbb{R}^3$, apply innocuous operations (projection-complement-projection), and get a non-measurable subset of $\mathbb{R}$.

Forcing and the Independence of the Continuum Hypothesis. Introduced by Paul Cohen, *forcing* is the most powerful technique of proving independence results. In a nutshell, it is a method of adjoining a new object to a given model of ZFC, analogous to adjoining, say, a root of $x^2 - 2$ to $\mathbb{Q}$: one gives this root a name, $\sqrt{2}$, and builds the field $\mathbb{Q}(\sqrt{2})$ as the set of all rational functions over $\mathbb{Q}$ and the name $\sqrt{2}$. However, building a new structure that satisfies the field axioms is *much* easier than building a new structure that is a model of ZFC, i.e. most of mathematics! To make the method transparent, we will not only present its traditional combinatorial treatment, but also its rephrasing in terms of Baire category. The course will culminate in the proof of the *Independence of the Continuum Hypothesis*.

PREREQUISITES: Familiarity with the basic concepts of mathematical logic (Math 570) such as structures, definability, and elementarity. A motivated student can quickly pick these up with the instructor's guidance, using, for example, the instructor's Mathematical Logic lecture notes available on her webpage.

REQUIRED WORK: Weekly homework with board presentations in weekly problem sessions.

EXAMS: None.